International Journal of Engineering Science 88 (2015) 64–72

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Simple shear of a compressible quasilinear viscoelastic material

Riccardo De Pascalis, I. David Abrahams, William J. Parnell *

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom

ARTICLE INFO

Article history:
Received 14 May 2014
Received in revised form 26 September 2014
Accepted 29 November 2014
Available online 23 January 2015

Keywords:
Quasilinear viscoelasticity
Hyperelasticity
Simple shear
Fung
Compressibility

ABSTRACT

Fung's theory of quasilinear viscoelasticity (QLV) was recently reappraised by the authors [Proc. R. Soc. A 479 (2014), 20140058] in light of discussions in the literature of its apparent deficiencies. Due to the utility of the deformation of *simple shear* in a variety of applications, especially in experiment to deduce material properties, here QLV is employed to solve the problem of the simple shear of a nonlinear compressible quasilinear viscoelastic material. The effects of compressibility on the subsequent deformation and stress fields that result in this isochoric deformation are highlighted, and calculations of the dissipated energy associated with both a 'ramp' simple shear profile and oscillatory shear are given.

1. Introduction

A number of materials that are capable of large deformations are also inherently *viscoelastic*, i.e. exhibit hysteresis under loading. Good examples are rubbers and other polymers (Simo, 1987; Abu Al-Rub, Tehrani, & Darabi, 2014) as well as soft biological tissues (Drapaca, Tenti, Rohlf, & Sivaloganathan, 2006; Johnson, Livesay, Woo, & Rajagopal, 1996; Peña, Calvo, Martínez, & Doblaré, 2007; Provenzano, Lakes, Corr, & Vanderby, 2002; Rashid, Destrade, & Gilchrist, 2012). In some circumstances such media can be modelled reasonably well as hyperelastic materials, using the theory of nonlinear elasticity, and their constitutive behaviour is defined by an associated strain energy function W. However in reality, and certainly when deformation rates are important, such a theory is not appropriate and a *nonlinear theory of viscoelasticity* is required. In particular an appropriate constitutive law has to be proposed, of which there are many as can be seen in the comprehensive review paper of Wineman (2009).

A popular approach is to use Fung's Quasilinear Viscoelasticity (QLV) theory, which assumes that viscous relaxation rates are independent of the instantaneous local strain. This model, which is a simplification of the Pipkin–Rogers model (Pipkin & Rogers, 1968) has been the subject of some criticism in recent years. However, in the authors' paper (De Pascalis et al., 2014) the law was recently reappraised and it was shown that such criticisms were unfounded, e.g. in a number of publications an incorrect QLV relation or stress measure was employed (the latter must be the second Piola–Kirchhoff stress to satisfy objectivity), or the incompressible limiting form was derived erroneously. QLV does of course have limitations; the fact that the relaxation functions are independent of strain is one of these. However the model appears to include enough detail to capture many of the essential elements of the physics whilst not being overly difficult to implement in the context of real-world applications.

* Corresponding author. Tel.: +44 161 275 5908.
E-mail addresses: riccardo.depascalis@gmail.com (R. De Pascalis), i.d.abrahams@manchester.ac.uk (I.D. Abrahams), william.parnell@manchester.ac.uk (W.J. Parnell).

http://dx.doi.org/10.1016/j.ijengsci.2014.11.011
0020-7225/

Of great value in a number of applications is the deformation of *simple shear*. In particular, the importance of this deformation to the biomechanical community is associated with the study of brain tissue for which stress measurements have been taken recently (Gilchrist, Rashid, Murphy, & Saccomandi, 2013; Rashid et al., 2012). For a general constitutive viscoelastic Pipkin–Rogers law, the stress–strain equations have been presented in Wineman (2009). This, and a number of other nonlinear viscoelastic models are, in practice, difficult to employ, and so Fung's QLV approach is attractive. However, all past applications of Fung's QLV model to simple shear are, to the authors' knowledge, deficient. It thus appears timely, from the viewpoint of both Fung's theory and applications in biomechanics and elsewhere, to study the classical problem of simple shear by employing the authors' new approach to QLV; this is therefore the aim of the present paper. We will carry out the full analysis of the simple shear problem for both incompressible *and* compressible viscoelastic materials of quasilinear type. Importantly, and unlike the case of simple shear of purely hyperelastic materials, even for isochoric deformations the effects of compressibility can have an influence on the deformation due to the memory effect of the relaxation term associated with hydrostatic compression.

The general form of the QLV constitutive equation can be written as

$$\mathbf{\Pi}(t) = \int_{-\infty}^{t} \mathbf{G}(t-s)\frac{\mathrm{d}\mathbf{\Pi}^{\mathrm{e}}}{\mathrm{d}s}(s)\ \mathrm{d}s, \tag{1}$$

where $\mathbf{G}$ is, in comparison with linear theory, the stress relaxation second-order tensor, $\mathbf{\Pi}$ denotes the second Piola–Kirchhoff stress, and t refers to time. The quantity $\mathbf{\Pi}^{\mathrm{e}}$ is a strain measure, which here can be considered as the *instantaneous* second Piola–Kirchhoff elastic stress response derived from a strain energy function W. As noted above, of fundamental importance in QLV is that the function $\mathbf{G}$ is independent of the local strain. The tensorial form (1) is a natural generalisation of the one-dimensional law proposed by Fung, 1981 and preserves objectivity.

It should be noted that many more complicated deformation states for compressible materials with QLV-type behaviour have been studied in the past, including those in Wineman and Waldron, 1995, Waldron and Wineman, 1996. However it appears that these and other papers, e.g. (Simo, 1987) incorporate only *one* relaxation function into the constitutive model for the material. For the general theory of isotropic compressible viscoelastic materials *two independent* relaxation functions are required in order for the constitutive behaviour to reduce to linear viscoelasticity upon taking the limit of small displacement gradients in the QLV constitutive law. A single relaxation function in the viscoelastic equation is only relevant to the case of *incompressible* materials as described in De Pascalis et al., 2014.

In Section 2 an overview of QLV in the context of compressible materials is given and the problem in hand here, i.e. simple shear, is stated. Note that timescales are assumed slow enough that the effects of inertia can be ignored. Explicit expressions are provided for the non-zero components of stress in terms of a general strain energy function. When the deformation is prescribed (i.e. for strain or displacement controlled experiments) this can simply be fed into these expressions to provide a prediction of the resulting stresses. On the other hand, in cases when the traction is prescribed these relations are nonlinear Volterra integral equations which must be solved for the resulting deformation (shear) field. Therefore, in Section 3 the procedure for solving the resulting integral equations is described, based upon the method introduced in De Pascalis et al., 2014. Since the above analysis applies to compressible materials, in Section 4 it is described how the various details are modified when the constraint of incompressibility, common in biomechanical and rubber-like material applications, is imposed. In Section 5 a number of results are given, associated with the simple-shear problem. It is shown that, even for this isochoric deformation, compressibility does have an effect on the resulting deformation and stress fields. The energy dissipated for a deformation that is piecewise linear in time is determined, as well as energy loss per unit cycle for oscillatory shear. We close in Section 6 with a brief summary and some directions for future research.

2. Constitutive laws and basic equations for simple shear

In the constitutive law (1), the elastic second Piola–Kirchhoff stress $\mathbf{\Pi}^{\mathrm{e}}$ under the integral is assumed hyperelastic, i.e. the instantaneous response can be derived from an energy potential W (the strain energy function). The deformation gradient tensor $\mathbf{F}$ is defined by

$$\mathbf{F}(s) = \begin{cases} \mathbf{I}, & s \in (-\infty, 0), \\ \dfrac{\partial \mathbf{x}}{\partial \mathbf{X}}(s), & s \in [0, t], \end{cases} \tag{2}$$

with $\mathbf{x}(s)$ denoting the position of a generic particle P at time $s \in [0, t]$, and $\mathbf{X}$ its position at the initial reference time. Note that the start time of the motion, and any imposed tractions, will be taken as $t = 0$. The quantity $J = \det \mathbf{F}$, expressing the local volume change, is a constant $J = 1$ when the deformation is isochoric. Let us assume that the material is isotropic and therefore from the deformation gradient tensor $\mathbf{F}$ we obtain the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and its principal invariants

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2\right] = (\det \mathbf{C})\mathrm{tr}(\mathbf{C}^{-1}), \quad I_3 = \det \mathbf{C} = J^2. \tag{3}$$

Given isotropy, the viscoelastic Cauchy stress is (referring to De Pascalis et al., 2014)

$$\mathbf{T}(t) = J^{-1}\mathbf{F}(t)\left(\mathbf{\Pi}^{\mathrm{e}}_{\mathrm{D}}(t) + \int_0^t \mathcal{D}'(t-s)\mathbf{\Pi}^{\mathrm{e}}_{\mathrm{D}}(s)\ \mathrm{d}s\right)\mathbf{F}^T(t) + J^{-1}\mathbf{F}(t)\left(\mathbf{\Pi}^{\mathrm{e}}_{\mathrm{H}}(t) + \int_0^t \mathcal{H}'(t-s)\mathbf{\Pi}^{\mathrm{e}}_{\mathrm{H}}(s)\ \mathrm{d}s\right)\mathbf{F}^T(t), \tag{4}$$

where $\mathbf{\Pi}_D^e$ and $\mathbf{\Pi}_H^e$ are the stress components of $\mathbf{\Pi}^e$ associated with the deviatoric and hydrostatic parts of the underlying instantaneous elastic Cauchy stresses, i.e.

$$\mathbf{\Pi}^e = \mathbf{\Pi}_D^e + \mathbf{\Pi}_H^e = J\mathbf{F}^{-1}(\mathbf{T}_D^e + \mathbf{T}_H^e)\mathbf{F}^{-T}, \tag{5}$$

with

$$\mathbf{T}_D^e = \mathbf{T}^e - \frac{1}{3}\mathrm{tr}(\mathbf{T}^e)\mathbf{I}, \quad \mathbf{T}_H^e = \frac{1}{3}\mathrm{tr}(\mathbf{T}^e)\mathbf{I}. \tag{6}$$

In terms of the strain energy we find

$$\mathbf{\Pi}_D^e = 2\left[\frac{1}{3}(I_2W_2 - I_1W_1)\mathbf{C}^{-1} + W_1\mathbf{I} - I_3W_2\mathbf{C}^{-2}\right], \tag{7}$$

$$\mathbf{\Pi}_H^e = 2\left(\frac{2}{3}I_2W_2 + \frac{1}{3}I_1W_1 + I_3W_3\right)\mathbf{C}^{-1}, \tag{8}$$

in which $W_j = \partial W/\partial I_j$ and $\mathcal{D}, \mathcal{H}$ are two scalar (independent) reduced relaxation functions (with $\mathcal{D}(0) = \mathcal{H}(0) = 1$). The latter relate to the inherent viscous processes associated with shear (deviatoric) and compressional (volumetric) deformations, respectively. Let us consider the homogeneous deformation of *simple shear* (see (De Pascalis, 2010) for an overview of the static nonlinear elastic case and related references),

$$x_1(t) = X_1 + k(t)X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{9}$$

where $k(t)$ is a time-dependent parameter representing the shear. The physical components of the deformation gradient tensor $\mathbf{F}$ and of its inverse $\mathbf{F}^{-1}$ are

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

respectively, so that the first three principal invariants of $\mathbf{C}$ are $I_1 = I_2 = 3 + k^2,\ I_3 = 1$.

Let us assume that $k(t)$ is such that inertial effects can be neglected; therefore this homogeneous deformation automatically satisfies the equations of motion (equilibrium). As stated earlier, this requires an assumption on the relationship between the shear and the relaxation timescales of the material. The resulting components of stress are, after some algebra, given by $T_{13}(t) = T_{23}(t) = 0$ and

$$\begin{aligned} T_{11}(t) = {} & 2((k^2(t)+1)W_1(t) + (k^2(t)+2)W_2(t) + W_3(t)) - \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k(s)[W_1(s)(k(s)((k(s)-k(t))^2+4) - 6k(t)) \\ & + W_2(s)(k(s)(2(k(s)-k(t))^2+5) - 6k(t))]\,\mathrm{d}s + \frac{2}{3}\int_0^t \mathcal{H}'(t-s)((k(s)-k(t))^2+1)[(k^2(s)+3)(W_1(s)+2W_2(s)) \\ & + 3W_3(s)]\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} T_{22}(t) = {} & 2(W_1(t) + 2W_2(t) + W_3(t)) - \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k^2(s)(W_1(s)+2W_2(s))\,\mathrm{d}s + \frac{2}{3}\int_0^t \mathcal{H}'(t-s)[(k^2(s)+3)(W_1(s) \\ & + 2W_2(s)) + 3W_3(s)]\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} T_{33}(t) = {} & 2((k^2(t)+2)W_2(t) + W_1(t) + W_3(t)) + \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k^2(s)(W_2(s)-W_1(s))\,\mathrm{d}s + \frac{2}{3}\int_0^t \mathcal{H}'(t-s)[(k^2(s) \\ & + 3)(W_1(s)+2W_2(s)) + 3W_3(s)]\,\mathrm{d}s, \end{aligned} \tag{11}$$

and

$$\begin{aligned} T_{12}(t) = {} & \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k(s)[W_1(s)(k^2(s)+3-k(s)k(t)) + W_2(s)(2k(s)(k(s)-k(t))+3)]\,\mathrm{d}s - \frac{2}{3}\int_0^t \mathcal{H}'(t-s)(k(s) \\ & - k(t))[(k^2(s)+3)(W_1(s)+2W_2(s)) + 3W_3(s)]\,\mathrm{d}s + 2k(t)(W_1(t)+W_2(t)). \end{aligned} \tag{12}$$

As in the nonlinear elastic case *both* normal and shear stresses are present on surfaces parallel to the coordinate planes and in this viscoelastic case they are time dependent (De Pascalis, 2010). Moreover, in the case when $\mathcal{D}' \equiv \mathcal{H}' \equiv 0$, it is easy to show that the *universal relations* for isotropic nonlinear elastic solids,

$$T_{13}(t) = T_{23}(t) = 0, \quad k(t)T_{12}(t) = T_{11}(t) - T_{22}(t), \tag{13}$$

hold. However, in analogy with the simple shear equations for a Pipkin–Rogers constitutive law (Wineman, 2009), here the integral terms do not allow (13) to be satisfied and Eq. (12) is odd in the shear, i.e. if k is replaced by $-k$, then the stress T_{12} is replaced by $-T_{12}$.

3. Solutions of the integral equations

In the simplest case, if we impose the *deformation* $k(t)$ (assuming that appropriate relaxation functions have been chosen and given a specific strain energy function W associated with the instantaneous effective elastic stress) then the equations (11) immediately determine all the components of stress. Physically, however, it is more common to impose tractions and so, in this case, Eqs. (11) are of the nonlinear integral Volterra-type, to be solved for the unknown deformation $k(t)$. Analytical solutions are not possible in general for this viscoelastic deformation. As such it is convenient to implement the numerical procedure proposed in the appendix of De Pascalis et al., 2014. There attention was restricted to *incompressible* materials and so here the procedure is modified for the compressible case. It is convenient to write the Volterra integral equations in the following separable form

$$T(t) = g(k(t)) + \sum_{i=1}^{2}\sum_{j=1}^{N} f_{ij}(k(t)) \int_0^t \mathcal{G}_i'(t-s) h_{ij}(k(s))\, \mathrm{d}s, \tag{14}$$

where $T(t), \mathcal{G}_i(t), g(X), f_{ij}(X)$ and $h_{ij}(X)$ are all known functions of their respective arguments. Of specific interest, due to its association with a common experiment, is when the shear stress component T_{12} is imposed, i.e. $T(t) = T_{12}(t)$ in (14). With this, and referring to (12), set

$$\mathcal{G}_1 = \mathcal{D},\ \mathcal{G}_2 = \mathcal{H}, \quad g(k(t)) = 2k(t)(W_1(t) + W_2(t)),$$

together with

$$f_{11}(k(t)) = k(t), \quad f_{12}(k(t)) = 1, \quad f_{21}(k(t)) = k(t), \quad f_{22}(k(t)) = 1,$$

and

$$\begin{aligned}
h_{11}(k(s)) &= -\frac{2}{3}k^2(s)(W_1(s) + 2W_2(s)),\\
h_{12}(k(s)) &= \frac{2}{3}k^3(s)(W_1(s) + 2W_2(s)) + 2k(s)(W_1(s) + W_2(s)),\\
h_{21}(k(s)) &= \frac{2}{3}(k^2(s) + 3)(W_1(s) + 2W_2(s)) + 2W_3(s),\\
h_{22}(k(s)) &= -\frac{2}{3}k(s)(k^2(s) + 3)(W_1(s) + 2W_2(s)) - 2k(s)W_3(s).
\end{aligned} \tag{15}$$

As details of the numerical procedure were given in De Pascalis et al., 2014 we do not dwell on numerical aspects here. This discussion aims only to illustrate and compare some aspects of the theory of quasi-linear viscoelasticity in the context of simple shear. In particular, we will carry out (numerical) cycling experiments in order to understand the influence of material properties and relaxation times on the internal rate of working, as is described in the next section.

To illustrate results, we need a choice of strain energy function; as such we shall employ the compressible Neo-Hookean model proposed by Levinson and Burgess, 1971

$$W = \frac{\mu}{2}(I_1 - 3) + \left(\kappa + \frac{\mu}{3}\right)(I_3 - 1) - 2\left(\kappa + \frac{4}{3}\mu\right)\left(I_3^{1/2} - 1\right), \tag{16}$$

where μ is the infinitesimal shear modulus and κ is the infinitesimal bulk modulus, and then

$$W_1 = \frac{\mu}{2}, \quad W_2 = 0, \quad W_3 = -\mu.$$

We shall consider the stress relaxation properties later.

4. Incompressible materials

When the material in question is considered to be *incompressible* the problem is posed somewhat differently to the above. In particular, the constitutive law has to be written down with care as I_3. For QLV it transpires that (4) reduces to De Pascalis et al., 2014

$$\mathbf{T}(t) = \mathbf{F}(t)\left(\mathbf{\Pi}_D^e(t) + \int_0^t \mathcal{D}'(t-s)\mathbf{\Pi}_D^e(s)\ \mathrm{d}s\right)\mathbf{F}^T(t) - p\mathbf{I}, \tag{17}$$

where $p(t)$ is the Lagrange multiplier required to satisfy the constraint of incompressibility. Eq. (7) becomes

$$\boldsymbol{\Pi}^{e}_{D}(t) = 2\left[\left(\frac{I_2}{3}W_2 - \frac{I_1}{3}W_1\right)\mathbf{C}^{-1} + W_1\mathbf{I} - W_2\mathbf{C}^{-2}\right]. \tag{18}$$

For a simple shear deformation, the components of stress are given by $T_{13}(t) = T_{23}(t) = 0$ and

$$\begin{aligned}
T_{11}(t) &= \frac{2}{3}(2W_1(t) + W_2(t))k^2(t) - p(t) - \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k(s)[W_1(s)(k(s)(4 + (k(s) - k(t))^2) - 6k(t)) \\
&\quad + W_2(s)(k(s)(5 + 2(k(s) - k(t))^2) - 6k(t))]\,\mathrm{d}s, \\
T_{22}(t) &= -\frac{2}{3}(W_1(t) + 2W_2(t))k^2(t) - p(t) - \frac{2}{3}\int_0^t \mathcal{D}'(t-s)(W_1(s) + 2W_2(s))k^2(s)\,\mathrm{d}s, \\
T_{33}(t) &= \frac{2}{3}(W_2(t) - W_1(t))k^2(t) - p(t) - \frac{2}{3}\int_0^t \mathcal{D}'(t-s)(W_1(s) - W_2(s))k^2(s)\,\mathrm{d}s, \\
T_{12}(t) &= 2(W_1(t) + W_2(t))k(t) + \frac{2}{3}\int_0^t \mathcal{D}'(t-s)k(s)[W_1(s)(3 + k^2(s) - k(s)k(t)) + W_2(s)(3 + 2k(s)(k(s) \\
&\quad - k(t)))]\,\mathrm{d}s.
\end{aligned} \tag{19}$$

As for the compressible case, if $k(t)$ is specified then the time-dependent tractions can be determined directly from these equations. Alternatively, the usual physical situation is that the tractions are imposed. As such T_{12} would be given and $k(t)$ determined from the nonlinear Volterra integral equation, via the numerical scheme referred to above and given in the appendix of De Pascalis et al., 2014. The Eq. (19) for shear stress is written as

$$T_{12}(t) = g(k(t)) + \sum_{j=1}^{N} f_j(k(t))\int_0^t \mathcal{D}'(t-s)h_j(k(s))\,\mathrm{d}s, \tag{20}$$

where $T_{12}(t), \mathcal{D}(t), g(X), f_j(X)$ and $h_j(X)$ are all known functions of their respective arguments and now here we have only *one* relaxation function ($i = 1$ in the expression (14)) as opposed to two in the compressible case (14). In particular, equating (20) with (19) we must choose

$$\begin{aligned}
&g(k(t)) = 2(W_1(t) + W_2(t))k(t), \quad f_1(k(t)) = k(t), \quad f_2(k(t)) = 1, \\
&h_1(k(s)) = -\frac{2}{3}k^2(s)(W_1(s) + 2W_2(s)), \\
&h_2(k(s)) = \frac{2}{3}k(s)[(k^2(s) + 3)W_1(s) + (2k^2(s) + 3)W_2(s)].
\end{aligned}$$

As regards the remaining stress components, given that $k(t)$ is now determined, imposing one of the normal components of stress T_{11}, T_{22} or T_{33} yields the function $p(t)$ and it is then possible to solve for the remaining fields.

A plethora of strain energy functions could be chosen when evaluating the deformation. Two of the most common forms to be employed are the neo-Hookean model

$$W^{\mathrm{NH}} = \frac{1}{2}\mu(I_1 - 3), \tag{21}$$

and Mooney–Rivlin strain energy function

$$W^{\mathrm{MR}} = \frac{1}{2}\mu[C_1(I_1 - 3) + (1 - C_1)(I_2 - 3)]. \tag{22}$$

Note that the neo-Hookean form is recovered from Mooney–Rivlin when the constant C_1 is set to the value unity. In Section 5 we choose, for simplicity, to employ the neo-Hookean form for an incompressible material.

5. Predicted deformation, stress and dissipated energy

Let us now consider the shear deformation, or stress, that results from the imposition of a specified shear profile. We begin in Section 5.1 by considering the influence of compressibility in the simple shear problem, where we increase and then decrease the deformation, or stress, linearly in time. In contrast to the perfectly elastic case, here we note that the memory effect of compressibility plays a role even for this isochoric deformation. In Section 5.2 we then determine the dissipation of energy associated with an imposed linear deformation, and with the more complicated field that results when oscillations are superposed on an initial linear shear. In the latter case we determine the energy dissipated over one cycle, after significant time, so that a steady state has been reached.

To specify matters, let us consider a material with moduli ratios (in the compressible case)

$$\frac{\mu_\infty}{\mu} = M = 0.4, \quad \frac{\kappa_\infty}{\kappa} = K = 0.8, \quad \frac{\kappa}{\mu} = 100, \tag{23}$$

where clearly only the first needs to be specified in the incompressible case. We note that μ and μ_∞ are respectively the infinitesimal shear modulus and the long-time (elastic) shear modulus. Similarly, κ and κ_∞ are the infinitesimal bulk modulus and the long-time (elastic) bulk modulus, respectively. Let us choose the stress relaxation functions to be classical one-term Prony series of the form

$$\mathcal{D}(t) = M + (1-M)e^{-t/\tau_d}, \qquad \mathcal{H}(t) = K + (1-K)e^{-t/\tau_h}, \tag{24}$$

where τ_d and τ_h are the associated deviatoric and compressive relaxation times. Note that we are always able to scale time on one of these relaxation times and we shall indeed do this shortly by plotting results as a function of t/τ_d. Hence, we need only specify the relaxation time ratio $\tau = \tau_d/\tau_h$, noting that as $\tau \to \infty$ the material becomes more elastic in its hydrostatic response (typical for rubber-like materials for example). As it will be seen shortly, in the isochoric deformation case considered here this limit loosely corresponds to the case of incompressibility.

We illustrate the responses below in the case of three different materials. A perfectly incompressible neo-Hookean viscoelastic material with strain energy function as defined in (21) and a compressible Levinson–Burgess material with strain energy function as defined in (16). In the former (indicated by a solid line in the forthcoming figures) we are not required to specify τ of course, whereas in the latter we consider the two cases $\tau = 1$ and $\tau = 10$ (indicated in the figures by dotted and dashed lines respectively).

5.1. Influence of compressibility on the isochoric shear deformation

As described above, since the simple shear deformation (9) is isochoric, in the perfectly elastic problem compressibility plays no role in the solution. In contrast, in the viscoelastic problem it does, by virtue of the memory effect of the compressive relaxation function $\mathcal{H}(t)$. We can illustrate this by imposing the simple shear (9) with $k(t)$ defined by

$$k(t) = \begin{cases} A_0 t/\tau_d, & t \in [0, t_*/2], \\ A_0(t_* - t)/\tau_d, & t \in [t_*/2, t_*], \\ 0, & t > t_*. \end{cases} \tag{25}$$

for some positive A_0. This is illustrated in Fig. 1 (a) with $t_*/\tau_d = 8, A_0 = 0.35$ and the resultant (scaled) shear stress $T_{12}(t)/\mu$ is shown in 1(b). The response of the compressible Levinson–Burgess material with $\tau = 1$ is shown as the distinct dotted line. Note that the behaviour of the compressible material with $\tau = 10$ (dashed line) is almost indistinguishable from the neo-Hookean incompressible (solid line) curve. In Fig. 1(c) the experiment is repeated for an imposed ramp shear stress $T_{12}(t)/\mu$ given in the form (25) with $t_*/\tau_d = 8, A_0 = 0.35$. The resultant shear $k(t)$ is illustrated in figure (d) for the same three materials as above, and is computed using expressions (14) and (20). As can be seen, the gross response of the compressible and incompressible materials is similar, for both prescribed shear and stress, especially as the relaxation ratio τ becomes large.

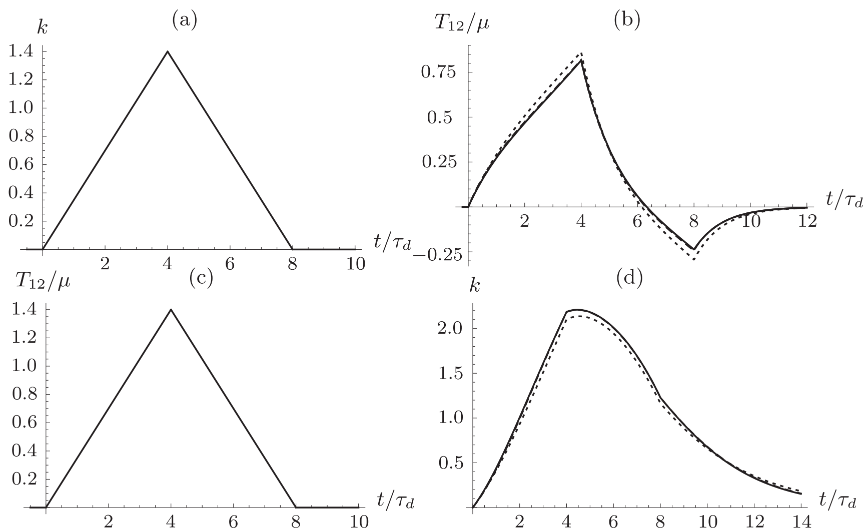

Fig. 1. In (a) we illustrate the imposed 'ramp' shear deformation $k(t)$. The resulting scaled stress T_{12}/μ is given in (b) in the case of the neo-Hookean incompressible material (solid) and the compressible Levinson–Burgess material with $\tau = 1$ (dotted line) and $\tau = 10$ (dashed line). The numerical experiment is repeated in (c) for an imposed stress, with resultant shear indicated in (d).

5.2. Dissipation of energy

We now consider the influence of the imposed deformation (or traction) and material response on the dissipation of energy due to the viscous nature of the material. Define the velocity gradient tensor by

$$\mathbf{D} = \frac{1}{2}\left(\dot{\mathbf{F}}\mathbf{F}^{-1} + (\dot{\mathbf{F}}\mathbf{F}^{-1})^T\right), \tag{26}$$

where the superposed dot represents the material time derivative. Since the deformation (9) is isochoric then $\mathrm{tr}\mathbf{D} \equiv 0$. The internal rate of working of the stress per unit current volume is $\mathbf{T}(t) : \mathbf{D}(t) \equiv \mathrm{tr}(\mathbf{T}(t)\mathbf{D}(t)) = T_{12}\dot{k}$. The (time-averaged) dissipated energy over a period t_* for the deformation imposed in (25) is therefore

$$E_d = \frac{1}{t_*}\int_0^{t_*} T_{12}(t)\dot{k}(t)\,\mathrm{d}t. \tag{27}$$

We can obtain a slightly modified version of the dissipated energy, suitable for determining the energy dissipated in oscillatory deformations over a period $\bar{t}$. Taking the start of the period as t_0, the time averaged dissipated energy is

$$E_d = \frac{1}{\bar{t}}\int_{t_0}^{t_0+\bar{t}} T_{12}(t)\dot{k}(t)\,\mathrm{d}t. \tag{28}$$

Clearly, if t_0 is chosen sufficiently large such that the initial transients have died out, then the deformation will have reached a steady state and E_d will be independent of t_0.

5.2.1. Single strain cycle

First, we consider the simple shear deformation as defined in (25). The important non-dimensional parameters are A_0 and t_*/τ_d which are respectively the rate of deformation and total time-scale of deformation, relative to the deviatoric relaxation time-scale. Note that we must be careful as to the magnitude chosen for A_0 since we have assumed from the outset that the deformation can be described without inclusion of inertial terms in the equation of motion. Fig. 2 offers the nondimensionalised energy dissipation as a function of A_0, for three different values of t_*/τ_d. Unsurprisingly, the larger the deformation, the larger the normalised dissipation. Note, as can be seen, the dissipation appears insensitive to the chosen material, except for large A_0 and $t_*/\tau_d = 5$, when the curve of the compressible material with $\tau = 1$ deviates from the other curves.

5.2.2. Periodic cycling superposed on a large deformation

The second case is a linear shear deformation, growing in time, with a superposed oscillatory shear, i.e.

$$k(t) = \begin{cases} A_0 t/\tau_d, & t \in [0, t_*] \\ A_0 t_*/\tau_d + A\sin[\omega\tau_d(t - t_*)/\tau_d], & t \in (t_*, \infty). \end{cases} \tag{29}$$

where A/A_0 controls the magnitude of the oscillation relative to the magnitude of the initial shear, and $\omega\tau_d$ is the (non-dimensional) frequency of oscillation relative to the deviatoric relaxation time. We stress once again that inertial effects have been ignored. Therefore in this application we must not choose the magnitude of $\omega\tau_d$ (the non-dimensional frequency) too large; as this parameter increases, inertial effects will become more important. Imposing (29) with $A = 1, A_0 = 0.6$ and $\omega\tau_d = 16\pi$ yields the displacement and subsequent shear stress fields as illustrated in Fig. 3. To avoid confusion, in this fig-

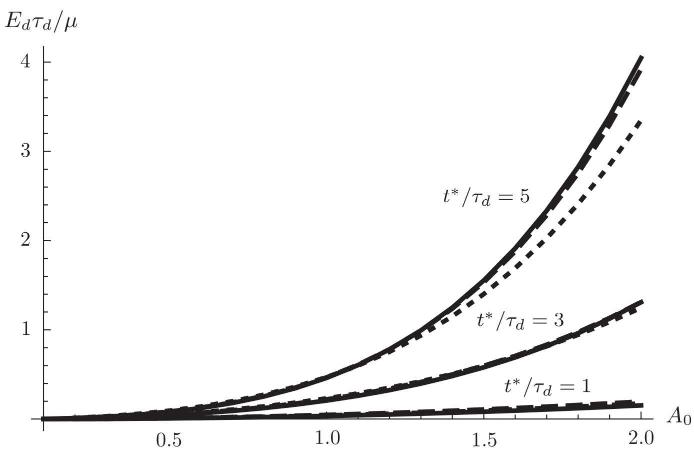

Fig. 2. Plot of the normalised dissipated energy against slope of the shear ramp profile for three different values of t^*/τ_d. The solid curves are for the incompressible material, dashed (dotted) curves are the compressible calculations for $\tau = 10$ ($\tau = 1$).

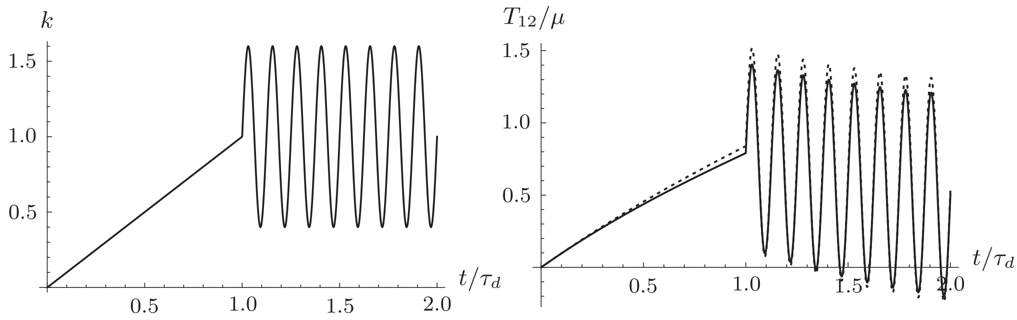

Fig. 3. Calculated shear stress response to the indicated shear $k(t)$, given by (29), with $A = 1, A_0 = 0.6, t^*/\tau_d = 1$, and $\omega\tau_d = 16\pi$. As previously, the solid curve is for a neo-Hookean incompressible material and the dotted line is the Levinson–Burgess compressible prediction with $\tau = 1$.

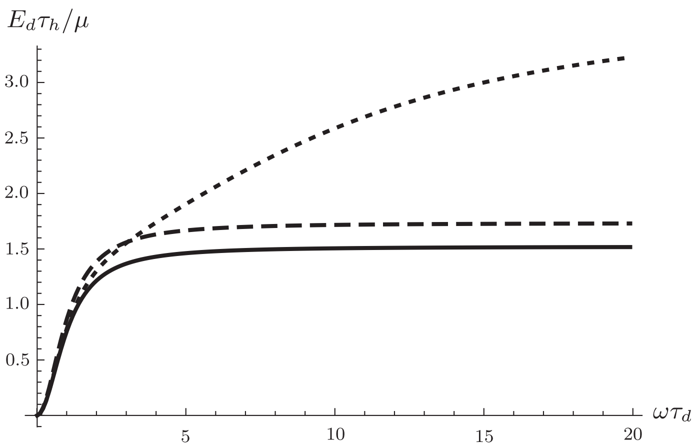

Fig. 4. Normalised long-time dissipated energy per unit cycle (28) for the shear profile, $k(t)$, given by (29) with $A_0 = 1, A = 0.6, t_*/\tau_d = 1$. The solid curve is for the incompressible material, dashed (dotted) for a compressible material with $\tau = 0.1 (\tau = 1)$.

ure only the compressible material with the relaxation-time ratio $\tau = 1$ is plotted alongside the incompressible result. The long-time dissipated energy over a single cycle is shown in Fig. 4 for a range of frequencies ω. It can be clearly observed that the energy dissipated by the compressible material is greater than that for the neo-Hookean solid. The energy dissipation appears constant at moderate and higher oscillation frequencies, except for the case with $\tau = 1$, which only becomes constant at very high frequencies. The frequency independence of the dissipated energy at higher frequencies appears to be consistent with other theory, e.g. Fig. 10 of Lion, 1997 where, for time-harmonic oscillations superposed on an initial deformation, plots of the temperature against frequency are plotted. One set of curves in that figure does not level off for the frequency range considered but we expect that at higher frequencies, beyond the range plotted, constant values will be achieved.

6. Conclusions

This paper has shown that the revised form of Fung's QLV model, proposed recently by the authors (De Pascalis et al., 2014), offers an effective and efficient way to model nonlinear viscoelastic materials undergoing simple-shear deformation. The model is able to incorporate a wide range of behaviours through the choice of instantaneous strain measure (modelled via an effective hyperelastic stress and underlying strain energy function) and relaxation functions. In this paper we examined two material models, one incorporating incompressibility proposed by Levinson and Burgess, 1971 and the other a neo-Hookean (incompressible) material, and chose to take a simple one-term Prony series to account for the fading memory of the deformation history. It was further assumed that rates of deformation are slow enough that inertial effects can be neglected; hence, as the deformations are spatially homogenous they automatically satisfy equilibrium.

The major simplifying assumption of QLV is that the relaxation functions are independent of the strain. This may lead to inaccuracies with some types of materials, but can be expected to offer a reasonable model for many practical purposes, such as when determining small perturbations about a large deformation, e.g. waves on a pre-stressed body. Separating the relaxation function from the strain measure in the Boltzmann superposition integral allows one to obtain an explicit relation between the viscoelastic stress $\boldsymbol{T}$ and the strain, or in the present case, the simple shear $k(t)$. For the models employed herein, the relations are given explicitly in Eqs. (12) and (19), which are easily evaluated to find $T_{12}(t)$ for a given $k(t)$. When the stress profile is specified then these equations offer nonlinear Volterra integral equations to solve for k(t), which are easily evaluated numerically as described in Sections 3 and 4.

The numerical results, discussed in the previous section, offer a number of points. First, whether the shear deformation (strain) or shear stress is imposed, the results obtained by our model appear physically 'reasonable' (see Figs. 1 and 3). Second, even though simple shear is isochoric (instantaneously volume preserving), material compressibility has an effect through the memory of the past deformation. In general though, for the parameter values chosen in this article, the difference between compressible and incompressible materials is small except at large shear values (see Fig. 2). Third, the effect of compressibility is found to diminish as the ratio of relaxation times $\tau(=\tau_d/\tau_h)$ increases, although, as a fourth point, the energy dissipated over a forcing cycle is found to be greater for a compressible material than the incompressible material (see Fig. 4). Finally, the last figure reveals that the long-time dissipation over a single cycle increases monotonically at low frequencies but tends to a constant value at mid to high frequencies.

The strength of the present model is its relative simplicity, so that it can be applied to inhomogeneous deformations. The authors are currently applying the new method to the viscoelastic deformation around voids in rubber-like bodies subjected to time-varying hydrostatic loading, where equilibrium has to be enforced as an extra constraint. The same approach is also being utilised to study deformations of viscoelastic soft biological tissues, where insight can be gained as to the likely effect of large stretch, impact or other trauma.

Acknowledgements

The authors are grateful to the Engineering and Physical Research Council (EPSRC) for the award (Grant No. EP/H050779/1) of a postdoctoral research assistantship for De Pascalis, and to the Royal Society for a Wolfson Research Merit Award (2013–2018) for Abrahams. The work was partially undertaken under the Thales UK SMART hub agreement.

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